Review Article on Method for Building Minimal 3-semiregular Graphs

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Abstract: This endeavor has created minimum 3-semiregular graphs based on a measure that is specified as the graph's index. The graph's index is defined as the product of the number of vertices and edges in the graph. The 3-barbell graph, with index 15, is a 3-semiregular graph, while the n-barbell graph is n-semiregular. This is the 3-semiregular graph's minimal index. It has been determined in this work that there are just two 3-semiregular graphs of index16.

Keywords: semiregular graphs, barbell graphs.

1. Introduction

A graph with the same degree at every vertex is called a regular graph. Every vertex in an r-regular graph has degree r. The features of linked regular graphs are intriguing. Constructing an r-regular graph with a particular order and size is not always feasible. Specifically, there isn't an odd regularity graph with an odd number of vertices. A disconnected 1-regular graph is also present. A semiregular graph is one in which every vertex is located precisely two distances from every other vertex. The graph is referred to be an n-semiregular graph if this precise number is n. An method was created by Alison Northup [1] to ascertain the graph's n-semiregularity. He established that all vertex transitive graphs are n-semiregular and classified 2-semiregular graphs.

A graph is said to be $\frac{1}{d}-$ regular if every vertex has same number of vertices at distance d. If every vertex in the graph has the same number of vertices at every feasible distance, the graph is referred to as distance degree regular (DDR). In graph theory, creating graphs with desired attributes is an intriguing field of study. The smallest 3-semiregular graphs have been generated in this attempt. It is evident that a three-semiregular network has to have a minimum of seven vertices and eight edges, or eight vertices and eight edges. The index of a graph is defined as the total of its edges and vertices. The 3-semiregular nonbarbell graph's index is therefore larger than or equal to 16. One might turn to [2, 3] for fundamental topics in graph theory.

2. A few common n-semiregular graphs

Several common graphs, or n-semiregular graphs, are shown in this section.

(i) Every cycle C_n , $n \geq 3$ is 2-semiregular and C_2, C_3 are 0-semiregular graphs. Every complete graph is 0-semiregular and the complete bipartite graph $K_{m,n}$ is semiregular iff, $m = n$.

(ii) The Wagner graph has four semiregularities, whereas the Peterson graph has six.

(iii) A graph obtained from P_1 by connecting n other vertices to each of the vertices of P_1 is called n-barbell graph and the n-barbell graph is n-semiregular.Thus, for every $n \geq 0$ there exists an n-semiregular graph.

(iv) Let r_1, r_2,…r_k be the reduced residue system modulo m. A graph G with vertex set {v_0, v_1, v_2,…v_{m-1}} in which v_i is adjacent to v_j if and only if i - j (mod m) = r.n , for some n satisfying 1 ≤ n ≤ k, is called circulant graph corresponding to the integer m.The circulant graph is α-semiregular if m is odd where αis the number of integers not relatively prime to m, $\left(\frac{m-2}{2}\right)$-semiregular if m is even and 0-semiregular if m is prime.

(v) A graph operator $'\cdot'$ is called closure with respect to semiregular graphs if the graph obtained from two n-semiregular graphs by connecting vertices from one graph into another is also an n-semiregular graph. The one-one and onto mapping defined between set of vertices of two copies of $k_n$is a closure operator and the resulting graph is (n-1)-semiregular.

3. A feature of orderliness in graphs

A graph's structure is mostly dictated by the quantity of vertices and edges present. As a result, the index of the graph is defined as the total of the vertices and edges. In this case, the graphs are compared using the measure-index. Density of a graph is a comparable type of metric that has already been established in the literature. Since the density is frequently a rational number, the graph's index, which is always an integer, is extremely helpful for quantitatively comparing the graphs. A graph G_1 is said to be larger than the graph G_2 , if the index of G_1, denoted as I(G_1) is greater than the index of G_2. Consequently, two graphs are said to be equivalent quantitatively iff they have the same index. Interestingly, no path is equivalent quantitatively to any cycle and vice versa, as it can be observed that$I(C_n) = 2n$ and $I(P_n) = 2n + 1$. The following are some simple observations:

i) The smallest 1-semiregular graph is P_4 with index 7

ii) The smallest 2-semiregular graph is C_5 with index 10.

iii) The n-barbell graph is the smallest n-semiregular graph for $n = 3, 4, 5$. with indexes 15, 19, 23 respectively.

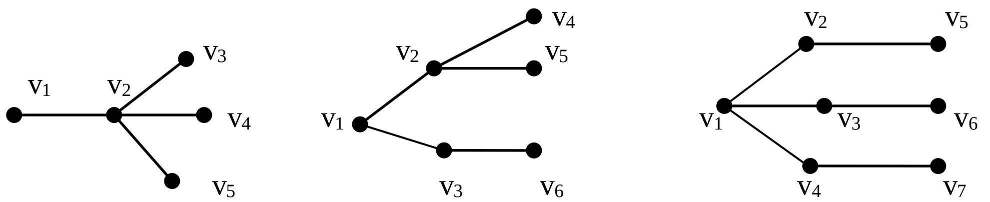
iv) With index 25, the Peterson graph is the smallest 6-semiregular graph. The 6-barbell graph's index is 27.

v) The index of n-barbell graph is 4n+3, n = 1,2,3,…

4. The minimum graphs of 3-semiregulars

One- and two-semiregular graphs were described by Alison Northup. If a simple linked graph is complete, it is 0-semiregular. The graph P_3, C_4 and a graph consisting finite number of components of P_3 or C_4 is 1-semiregular graph. Also, the complement of the union of finite number of path graphs P_1 is 1-semiregular.Additionally, Northup demonstrated that a connected graph is 2-semiregular if and only if it is an n-cycle, the complement of an n-cycle for n≥5, the complement of the union of a minimum of two disjoint cycles, or one of the seventeen graphs she picked from a list. This effort discusses graphs that are 3-semiregular, apart from the barbell graph with minimal index. It is clear that the 3-barbell graph, with index 15, is 3-semiregular. Therefore, we create the 3-semiregular graph with index 16 here. Any graph with index 16 should contain at least 6 vertices since k_5 has an index of 15.

Any one of the following graphs must be a subgraph of a 3-semiregular graph.



(i) (ii) (iii)

Fig. 1: Three-semiregular graphs' subgraphs

In the first graph one new vertex v_6 and 6 more edges to be added to get a 6-vertex graph with index 16. Since deg(v_2) = 4, at the most only the new vertex v_6 can be put at distance 2 from v_2. As a result, the first graph mentioned above cannot be the subgraph of a 3-semiregular graph. This likewise holds true for the second subgraph. Therefore, there isn't a 3-semiregular graph of order 6 with index 16. Now look at the third graph, which has size 6 and order 7. Therefore, out of the potential 15 edges, just 3 more edges need to be added in order to create a 7-vertex graph with index 16. There are 455 methods to go about this. Since adding a new edge from v_1 will alter the graph's 3-semiregularity, the new three edges should only be created between the six vertices that remain. i.e the new 3 edges can be only from the remaining 12 edges and totally there are 220 possible graphs, out of 455 graphs. The new 3-edges would have to give us that 1 vertex at distance 2 from each vertices v_2, v_3, v_4 and 2 vertices at distance 2 from each vertices v_5, v_6, v_7. Since v_2, v_3, v_4 are all of degree 2, only one edge is possible from these vertices. Hence there are only four routes available to obtain 3-semiregular graph. This is represented pictorially as follows

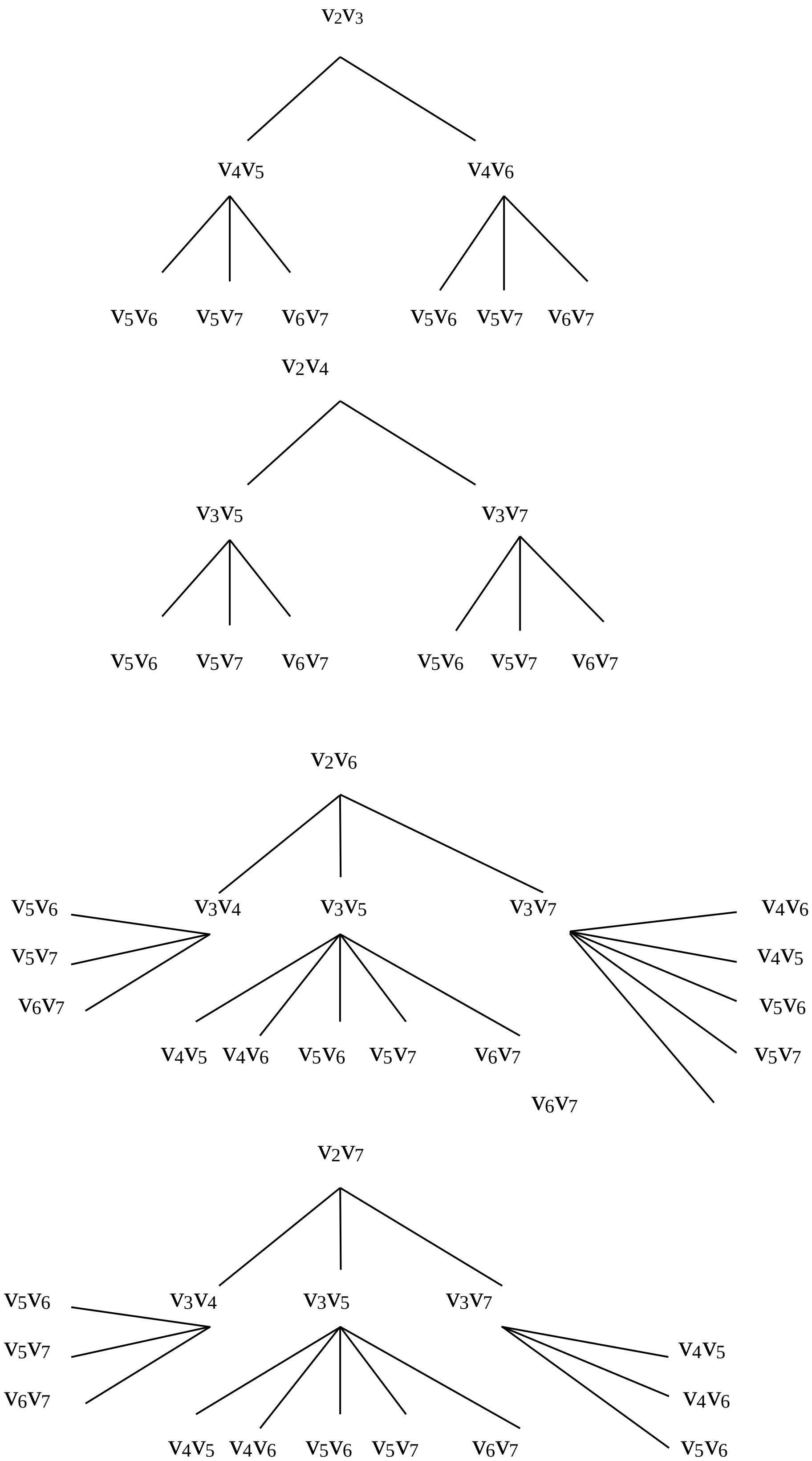


Fig. 2 A tree representation of edges

Thus, 36 different graphs are conceivable. The following graph is the only 3-semiregular

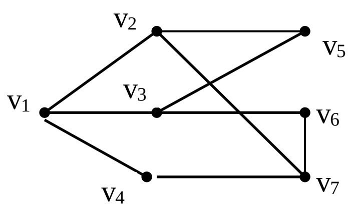
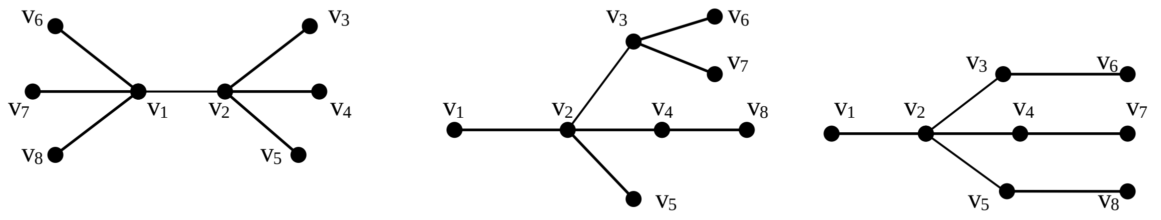


Fig.3 3-semiregular graph of index 16

The only nonbarbell 3-semiregular graphs of index 16 with more vertices that may be obtained are those with eight vertices and eight edges. Any of the graphs shown in Figure 1 should be a subgraph of these graphs.

To construct a 3-semiregular graph of index 16 from the first graph we have to add 3 more vertices and 4 more edges. Since, the vertex v_2 has no vertex at distance 2, all the new 3 vertices should be added so that the distance of these vertices from v_2 is 2. For this there are 20 possible graphs and if we discard the isomorphic graphs, we are left with only the 3 graphs given below



(i) (ii) (iii)

Fig. 4: Three-semiregular graphs

Among the 3-graphs above, the graph (i) is a barbell graph.In graph (iii) each of the vertices v_1, v_2, v_3, v_4, v_5 have 3 vertices at distance 2, whereas the vertices v_6, v_7, v_8 have only v_2 at distance 2 and one additional edge cannot give 3 vertices at distance 2 from v_6, v_7, v_8 . Therefore a 3-semiregular graph cannot be constructed from this graph. Graph (ii) becomes 3-semiregular by including the edge $v_3 v_8$ and the graph is given below

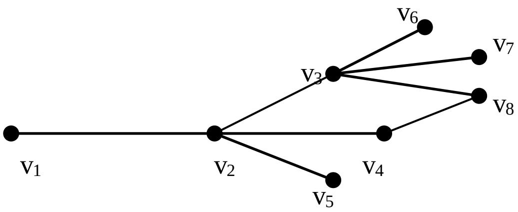


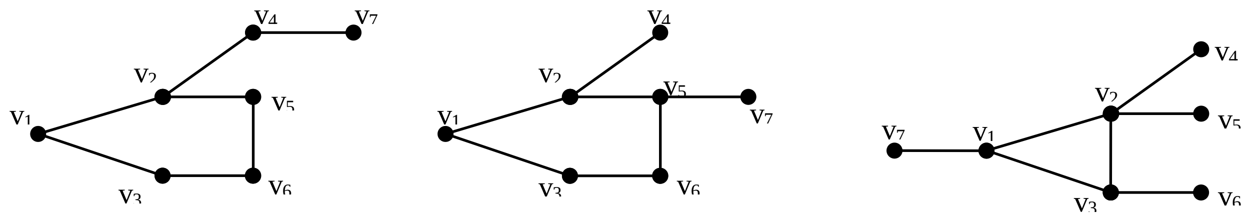
Fig. 5: Index 16 three-semiregular graphs

Now, let's construct a three-semiregular graph, where the graph in Fig. 1(ii) will be a subgraph. There are six vertices and five edges in this subgraph. Consequently, we examine how to build a graph three semiregular by adding two vertices and three edges. The vertices that are two distances away from one another are included in the following table along with possible vertices that might be added to make a three-semiregular.

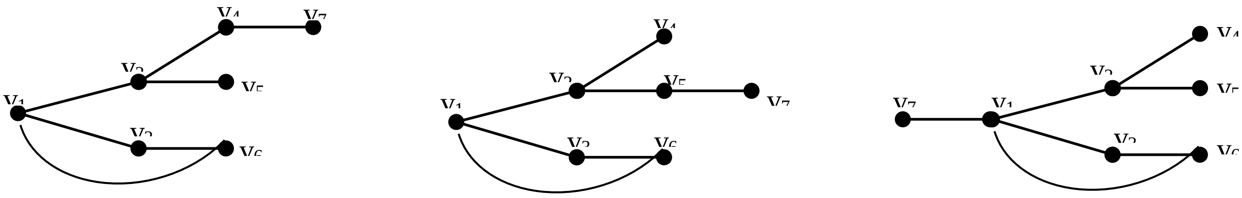
i.e., the new vertices at distance 2 from

vertex	Vertices at distance 2	Possible vertices to be at distance 2
v_1	v_4, v_5, v_6	-
v_2	v_3	v_6, v_7, v_8
v_3	v_2	v_4, v_5, v_7, v_8
v_4	v_1, v_5	v_3, v_6, v_7, v_8
v_5	v_1, v_4	v_3, v_6, v_7, v_8
v_6	v_1	v_2, v_4, v_5, v_7, v_8

$v_2$may be either v_6, v_7 or v_6, v_8 or v_7, v_8. If we choose v_6, v_7 then we get nine graphs as given below



(i) (ii) (iii)



(iv) (v) (vi)

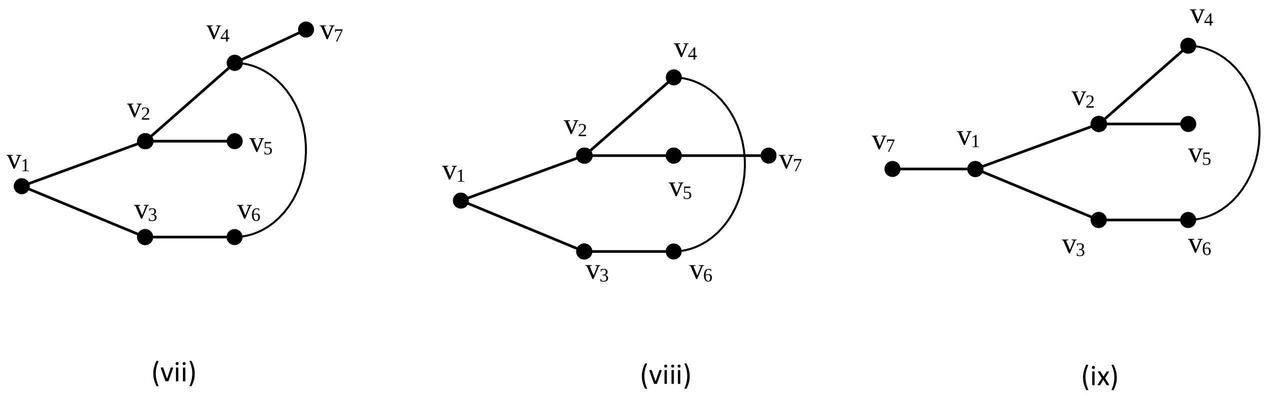


Fig. 6: Three-semiregular graphs

Which of the nine graphs will result in a 3-semiregular graph once the last vertex and edge are added?

Note that no graph contains three vertices at a distance of two from v_3, with the exception of graphs (iii) and (ix). There are two vertices at a distance of two from $v_3$in graphs (i), (ii), (vi), (vii), and (viii), but only one vertex at a distance of two from $v_3$in graphs (iv) and (v).

Hence, a new vertexv_8, which incidents with new edge, is to be added in the graphs (i), (ii), (vi), (vii) and (viii) so that the vertex v_8 becomes the third vertex at distance 2 from v_3. It can be seen that the new edge neither v_6v_8 nor v_1v_8 give the required 3 vertices at distance 2 from v_4, v_5, v_6, v_7, v_8 . Hence, 3-semiregular graph cannot be constructed from these graphs by adding an edge and a vertex.

In case of the graph (iii) by adding the edge $v_2v_8$we get three vertices v_1, v_5, v_8at distance 2 from v_4 and this is the only possible way of connecting $v_8$from v_2,so as to get three vertices at distance 2 from v_4 . But v_5 has 4 vertices v_1, v_3, v_4, v_8 at distance 2. Moreoverv_6 and v_7 have only 2 vertices at distance 2. Hence, a 3-semiregular cannot be constructed from this graph.

In graph (ix) v_4 also has 3 vertices v_1, v_5, v_3 at distance 2. So by adding the edge v_2v_8 we get three vertices v_1, v_4, v_8 at distance 2 from $v_5$and this is the only possibility to get 3 vertices at distance 2 from v_5 .But v_6 and v_7 have only 2 vertices at distance 2 and therefore a 3-semiregular cannot be constructed from this case.

In the remaining graphs, (iv) and (v), it is not possible to get 2 more vertices at distance 2 from v_3 as v_3 has only one vertex at distance 2. Thus by adding the edge v_1v_8 or v_6v_8 it is not possible to obtain a 3-semiregular graph.

So far, we have discussed the ways of constructing 3-semiregular graphs from the graph given in Fig. 1 (ii), so that the vertices v_6 and v_7 are at distance 2 from v_2. The above discussions are also applicable for the case of making v_6 and v_8 at distance 2 from v_2.

Therefore, let us investigate whether we can obtain a 3-semiregular graph by choosing the new vertices v_7, v_8 to be at distance 2 from v_2, for the same subgraph given in fig. 1 (ii). For this, the possible graphs are as follows

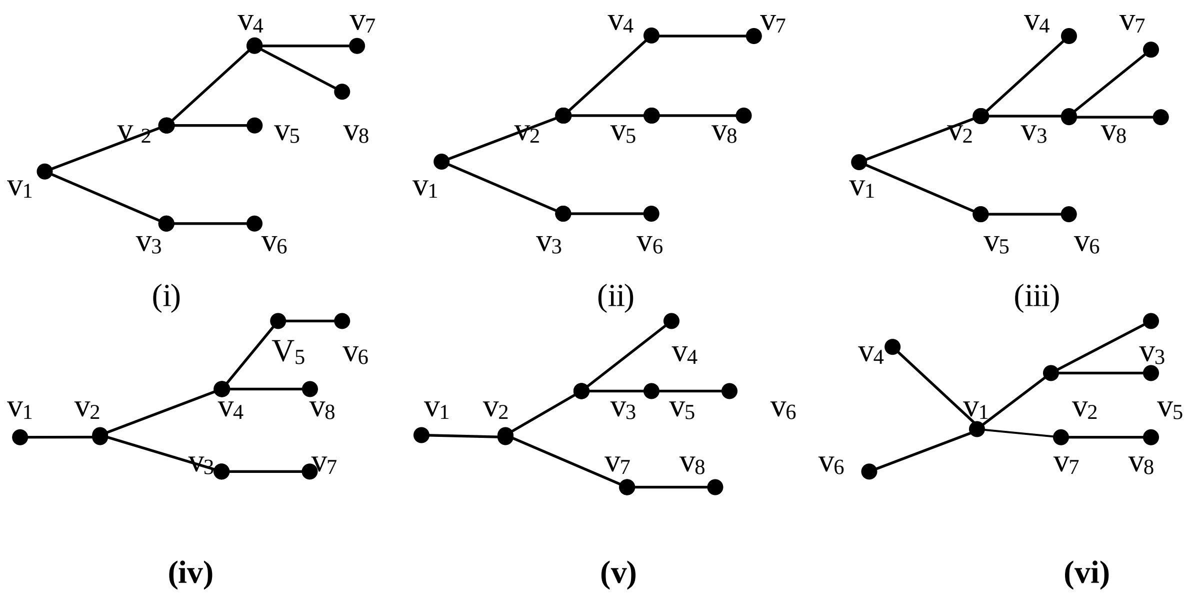


Fig. 7: Three-semiregular graphs

In fig. 7 (i), only the inclusion of the edgev_3v_4 give us 3 vertices at distance 2 from v_3. This edge also gives us 3 vertices v_1, v_5, v_6 at distance 2 from v_4. But v_5, v_6 have only 2 vertices at distance 2. Therefore, a 3-semiregular graph is not possible from this graph.

It should be noted that adding an extra edge to Figure 7(ii) does not provide us with three vertices at a distance of two from v_3. Therefore, this graph cannot be used to create a 3-semiregular graph.

In fig. 7 (iii), there is only one possibility of getting 3 vertices at distance 2 from v_3, by including the edge v_3v_5, but this is not giving us 3 vertices at distance 2 from v_4.

In graph (iv), there are three possible ways of including an edge to get 3 vertices at distance 2 from v_3, by adding the edges v_3v_4, v_3v_7, v_2v_3 respectively. In the first two cases, v_4 also have 3 vertices at distance 2 but v_5 has only 2 vertices at distance 2, whereas in the third case, v_4, v_5 have 3 vertices at distance 2, but v_6 has only 2 vertices at distance 2. Hence, the construction of 3-semiregular graph is not possible from this subgraph also.

In graph (v), there are two ways of getting 3 vertices at distance 2 from v_3 by including the edges v_3v_5 or v_3v_7. But, in both cases v_4 has only 2 vertices at distance 2. So, the possibility of getting 3-semiregular graph from this graph is ruled out.

In graph (vi), v_3 already has 3 vertices at distance 2. So we find ways of including edges to get 3 vertices at distance 2 from v_4. It can be done in four ways by including the edges v_2v_3, v_2v_7, v_2v_6 and v_2v_8 respectively. The inclusion of the edge $v_2v_3$in the graph makes all vertices to have 3 vertices at distance 2, except the vertex v_6. Similarly it can be found that the construction of 3-semiregular graph is not possible by including the edges v_2v_7, v_2v_8in the given graph. But a 3-semiregular graph is possible from this graph by including the edge v_2v_6. The graph is given below.

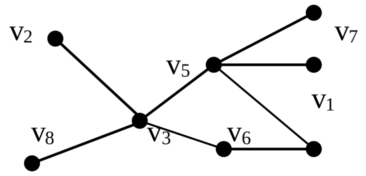


Fig. 8 3-semiregular graph with index 16

Lastly, have a look at the subgraph in fig. 1 (iii). There are 6 edges and 7 vertices in this graph. We simply need to add one new vertex and two more edges to create an 8-vertex graph with index 16. One of the two extra edges has to intersect the new vertex v_8. If we choose this edge as v_1v_8 then the vertices v_2, v_3, v_4 each will have 3 vertices at distance 2. The remaining one edge have to be chosen so that v_5, v_6, v_7 have 3 vertices at distance 2 which is impossible. To the maximum, we can bring one of these 3 vertices to have 3 vertices at distance 2 by choosing the edge from v_1 to the corresponding vertex. Hence, a 3-semiregular graph is not possible from this subgraph.

Therefore the possible 3-semiregular graphs with index 16 are

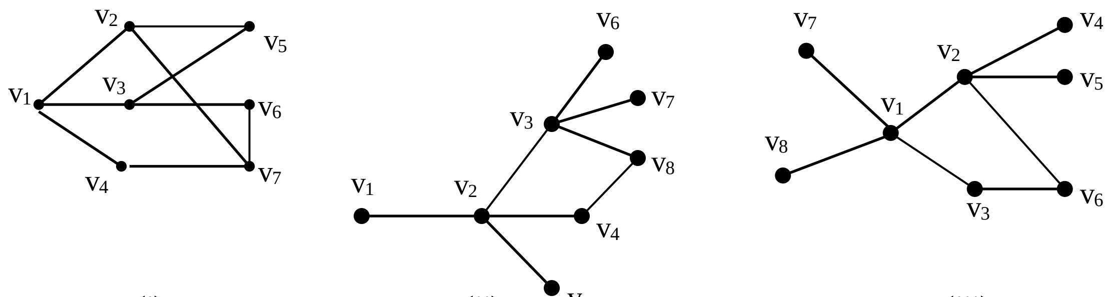


Fig.9 3-semiregular graphs with index 16

It should be noted that graphs (ii) and (iii) in Figure 9 are isomorphic, meaning they are the same but have different vertex names. Thus, as seen below, there are only two 3-semiregular graphs with index 16.

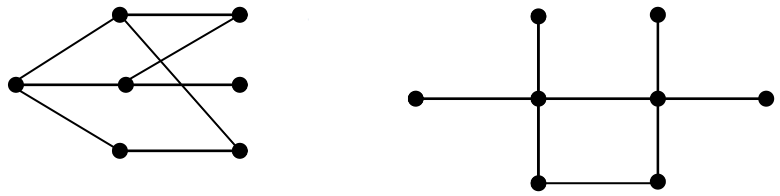


Fig. 10 3-semiregular graphs with index 16

5. Conclusion

In graph theory, creating graphs with desired properties is an intriguing topic. Almost all special graphs in graph theory, including the Wagner, Peterson, Franklin, and others, are generated and highlighted solely due to the distinctive properties included in their structural representations. Here, a comparable search for the existence of 3-semiregular graphs with minimal measure is conducted.

Reference

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